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Near-Frozen Quasi-One-Dimensional Flow. I. The Reservoir Problem

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NEAR-FROZEN QUASI-ONE-DIMENSIONAL FLOW
I. THE RESERVOIR PROBLEM

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CONTENTS

Non-equilibrium quasi-one-dimensional nozzle flows are considered in the limit when the relaxation time is large compared with some characteristic flow time. Non-uniformities which arise in the reservoir region, for convergent-divergent nozzles, are treated by the method of matched asymptotic expansions (see, for example, Van Dyke 1964). It is shown that even away from this stagnation zone the solution does not proceed simply in integral powers of the rate parameter. The correct solution is deduced for a vibrationally relaxing gas.

It is noted, however, that this near-frozen solution does not necessarily remain valid at downstream infinity where the overall entropy production may become important. Solutions valid in this region are presented in part II of this paper.

1. Introduction

The phenomenon of freezing in expanding non-equilibrium gas flows is now well documented and understood. A large number of numerical solutions have been obtained for various rate processes and in a few relatively simple cases it has proved possible to attack the problem analytically. Associated with these analytical solutions has been the realization that the sudden freezing point (Bray 1959) is a turning point of the governing differential equations (Blythe 1964*a*, *b*): this fact has been exploited, for example, in the limit when the rate parameter, or Damköhler number Λ, is large, to obtain the asymptotic frozen level of the energy in the lagging mode. (Λ is a characteristic ratio of the flow time to the time scale of the rate process.) The principal application of this approach has been to quasi-one-dimensional nozzle flows.

Not surprisingly, these solutions do not remain valid, for a convergent-divergent nozzle, if the freezing point (turning point) occurs sufficiently far upstream of the throat, which certainly implies that Λ is not large and it is in fact the limit $\Lambda \to 0$ that is considered in this paper. A distinction is made between these flows, which are termed near-frozen, and those governed by the existence of a freezing point at a finite station in the flow. The latter case is characterized by a near-equilibrium behaviour upstream of the freezing point (see figure 1 (*a*)). For a convergent-divergent nozzle, as $\Lambda \to 0$, the transition layer in the neighbourhood of the freezing point and the near-equilibrium region merge in the reservoir zone, where the flow time scale is locally 'large' (see figure 1 (*b*)). It is apparent that the usual description of the transition layer is inadequate in this case.

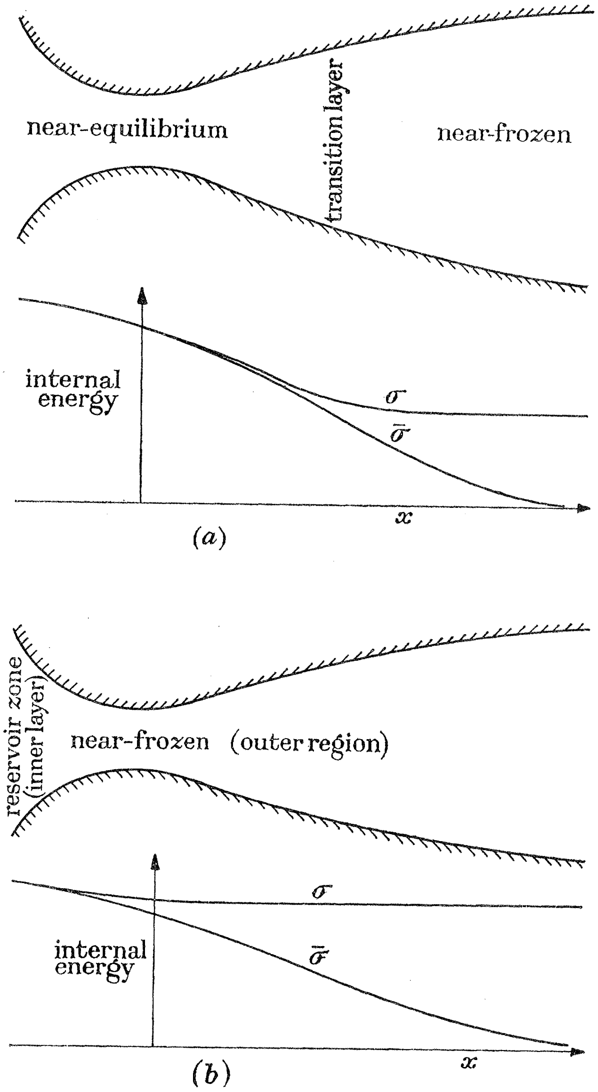

FIGURE 1. Flow regimes and characteristic internal energy distribution for (*a*) Λ large, (*b*) Λ small. σ is the vibrational energy and $\bar{\sigma}$ its local equilibrium value.

Near-frozen flows in which the velocity is everywhere non-zero have been considered elsewhere (see, for example, Bloom & Ting 1960). For such flows it is assumed that the solution can be expanded in integral powers of Λ. If this type of solution is to be applicable in the present context then it can only be downstream of the reservoir (or stagnation) zone, since a necessary condition on its validity is that the forcing term in the rate equation be

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small everywhere compared with the convective term. A demonstration that this approach does yield results which are singular at upstream infinity is given in § 3.

Within the reservoir zone the structure of the solution depends on the rate of growth of the nozzle area ratio A. The family of nozzles considered here are such that $A \sim |x|^n$ as $x \to -\infty$, where x is an appropriate non-dimensional distance.

For this class of nozzle shapes the convective and forcing terms are of the same order of magnitude when $|x| = O(\Lambda^{-1/(n+1)})$ (see § 4). The form of the equations simplifies considerably in this region. In particular, the zero-order approximations to the energy and velocity distributions are uncoupled and the problem reduces to solving a system of first-order linear differential equations in which the mass flow enters as an unknown parameter. This quantity depends on conditions at the sonic point, or critical point, defined by the frozen sound speed, the position of which lies outside the range of validity of the reservoir zone solution.

Downstream of the reservoir zone the behaviour is predominantly that of a frozen flow. The obvious approach is to assume that a conventional expansion in integral powers of Λ is valid in this region, and then attempt to determine the arbitrary constants that arise by matching the upstream behaviour of the near-frozen solution with the downstream expansion of the reservoir solution. However, from this downstream limit it is apparent that for $n \leqslant 1$ the first-order perturbation cannot be $O(\Lambda)$ in the near-frozen region (see § 5). For $n > 1$ the first-order perturbation is $O(\Lambda)$ but non-integral powers are present in higher order terms and the departure from the conventional near-frozen solution culminates with the occurrence of a term $O(\Lambda^2 \log \Lambda)$ (see §§ 6 and 7). As the mass flow rate is determined from this near-frozen solution it follows that ultimately this logarithmic term will be fed back into the reservoir solution.

An important extension of the approach outlined for the reservoir zone would be its application to the stagnation region in blunt body flows. Although such flows are not considered here it is interesting to note the limiting behaviour of an approximate solution to the problem. If the velocity is represented by its Newtonian value (Freeman 1958) the flow within the shock layer can be computed for arbitrary values of Λ. For a particularly simple rate equation (Blythe 1963) an expression for the stand-off distance in axially symmetric flow, for example, can be obtained in closed form. In the limit $\Lambda \to 0$ the discrepancy from the fully frozen result is not $O(\Lambda)$ but $O(\Lambda \ln \Lambda)$—a direct consequence of the stagnation layer near the body surface. The singular nature of the frozen limit for blunt body flows has also been discussed more recently by Conti (1966).

The analysis is outlined specifically for the de-excitation of the vibrational mode in a diatomic gas. Although the assumed rate equation (see § 2) for this process is simpler in form than the corresponding equations for dissociation, ionization, etc., its adoption involves no loss in generality for the solution within the reservoir zone. In that region, the departure from equilibrium is 'small' and the zeroth approximation to any rate equation is linear in terms of the non-equilibrium energy; consequently, the solution within the zone adopts a similar form for all rate processes (though differences of detail do arise in the algebra). Similar remarks can be made about the near-frozen solution for $x = O(1)$, since there the zeroth order solution is the fully frozen one for which the forcing term is identically zero.

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An interesting feature of the near-frozen solution is that it does not necessarily remain valid as $x \to +\infty$ even though the relaxation time τ becomes unbounded in this limit. If

$$\int^{x} \tau^{-1}\,dx \quad \text{diverges as} \quad x \to \infty$$

it appears that the internal energy σ in the inert mode will eventually decay to ground level, which is the ambient equilibrium state at downstream infinity. (This mode of behaviour can occur for arbitrary values of Λ; see Eschenroeder 1962.) However, the convergence of this integral is not sufficient to ensure the validity of the near-frozen solution for large x. It is also necessary that

$$\int^{\sigma(x)} \frac{d\sigma}{T}$$

is bounded in this limit. This latter integral is obviously associated with the entropy production. Indeed $\Delta\sigma$ can be specifically identified as an effective heat addition (Johannesen 1961). Constraints under which both these integrals are convergent are discussed in § 3. It is sufficient to note here, for the vibrationally relaxing gas considered, that if the 'entropy' integral is convergent then so too is

$$\int^{x} \tau^{-1} dx.$$

When the entropy integral is divergent the solution for large positive x has several interesting features. The most important of these would seem to be the possible existence of compression regions through which a relatively rapid de-excitation takes place and in which the temperature gradient is positive. Such regions are analogous to condensation shocks. Appropriate solutions in these cases will be presented for a vibrationally relaxing gas in part II of this paper.

A complete list of the notation used in both parts of this paper is given in appendix III at the end of part II.

2. The conservation and rate equations

The equations governing the quasi-one-dimensional flow of an ideal vibrationally relaxing gas (Johannesen 1961; Stollery & Park 1964) can be written in non-dimensional form as

$$\rho u A = m, \tag{2·1}$$

$$\frac{1}{\rho}\frac{d\rho}{dx} - \frac{1}{\gamma-1}\frac{1}{T}\frac{dT}{dx} = \frac{1}{T}\frac{d\sigma}{dx}, \tag{2·2a}$$

or

$$\rho T^{-1/(\gamma-1)} = \exp\left\{\int_{\sigma_\infty}^{\sigma} \frac{d\sigma}{T}\right\}, \tag{2·2b}$$

$$\frac{\gamma}{\gamma-1} T + \sigma + \tfrac{1}{2}u^2 = \frac{\gamma}{\gamma-1} + \sigma_\infty + \tfrac{1}{2}u_\infty^2, \tag{2·3}$$

$$u\frac{d\sigma}{dx} = \Lambda\rho\Omega(T)\,[\bar{\sigma}(T) - \sigma], \tag{2·4}$$

where ρ is the density, u the velocity, A the cross-sectional area at any station x, σ the vibrational energy, $\bar{\sigma}$ the local equilibrium value of σ corresponding to the translational temperature T, and Ω the relaxation frequency. γ is the specific heat ratio for the active modes of translation and rotation and m is a non-dimensional mass flow rate. The suffix ∞ denotes initial conditions where the flow is not necessarily in equilibrium. If, however, $u_\infty = 0$ then

$\sigma_\infty = \bar{\sigma}_\infty$. It is this particular case which corresponds to flow through a convergent-divergent nozzle.

The above non-dimensional variables are defined by

$$\left.\begin{aligned} &\rho = \frac{\rho'}{\rho'_\infty}, \quad T = \frac{T'}{T'_\infty}, \quad \Omega = \frac{\Omega'}{\Omega'_\infty}, \\ &\sigma = \frac{\sigma'}{RT'_\infty}, \quad u = \frac{u'}{\surd(RT'_\infty)}, \quad m = \frac{m'}{\rho'_\infty \surd(RT'_\infty) A'_t}, \\ &A = \frac{A'}{A'_t}, \quad x = \frac{x'}{h'}, \end{aligned}\right\} \tag{2·5}$$

where primes denote the dimensional quantity, R is the gas constant, h' is the minimum nozzle height (or radius), and the suffix t defines conditions at the minimum cross-section. Λ is the rate parameter defined by

$$\Lambda = \frac{\rho'_\infty \Omega'_\infty h'}{\surd(RT'_\infty)} = \surd\gamma \frac{h'/a'_\infty}{\tau'_\infty}, \tag{2·6}$$

where τ' is the relaxation time and a' is the frozen sound speed. The limit $\Lambda \rightarrow 0$ implies frozen flow and it is this situation which is of interest here.

It is assumed that the equilibrium distribution of the vibrational energy is given by the usual expression

$$\bar{\sigma} = \frac{\theta_v}{\exp\{\theta_v/T\} - 1}, \tag{2·7}$$

which holds for a system of harmonic oscillators. θ_v is the non-dimensional characteristic temperature of vibration.

A discussion of the validity of the rate equation (2·4) is beyond the scope of this paper. From the work of Shuler (1959) a necessary condition would appear to be that the fraction of excited oscillators should be small. However, it is usual, as here, to assume that the equation remains valid for all values of the translational temperature. Some evidence does exist to the contrary (Zienkiewicz & Johannesen 1963), but unfortunately no suitable alternative to (2·4) has yet been proposed. Any error in (2·4) is probably not important in the stagnation region where the flow is, in some sense, near to equilibrium and the true rate equation will presumably, to a first approximation, reduce to the form (2·4) with a suitable modification in the definition of τ'. For similar reasons the extension of the present analysis, within the stagnation zone, to the flow of a dissociating or ionizing gas is straightforward.

The relaxation frequency is in general a rather complex function of the translational temperature. Several expressions fit the experimental data reasonably well. Phinney (1964) correlates the experimental data for diatomic gases by means of an expression of the type

$$\Omega \propto T^\beta \exp\left\{-B\left(\frac{\theta_v}{T}\right)^{\frac{1}{3}}\right\}, \tag{2·8}$$

but Widom (1957) suggests that

$$\Omega \propto T^{\delta_0} \exp\left\{D_0\left(\frac{T}{\theta_v}\right)^{\frac{1}{3}}\right\}, \tag{2·9a}$$

or

$$\Omega \propto T^{\delta_1} \exp\left\{D_1 \frac{T}{\theta_v}\right\}. \tag{2·9b}$$

There is some theoretical justification for each of these expressions. Although they all fit the data fairly well over a reasonably broad temperature range, (2·8) and (2·9) have quite distinct behaviours at low temperatures, where available experimental results are insufficient to suggest which model is more likely to be appropriate in this limit. This difference may have important practical consequences on the asymptotic solution far downstream (see § 3 and part II). The analysis in part I will be outlined for a general $\Omega(T)$.

3. Singularities in near-frozen solutions

Near-frozen solutions, $\Lambda \to 0$, of equations (2·1) to (2·4) can be constructed via suitable expansions in Λ. The conventional expansion (Bloom & Ting 1960) is

$$\left.\begin{aligned} \sigma &= \sigma_\infty + \Lambda\sigma_1(x) + \Lambda^2\sigma_2(x) + \dots, \\ T &= T_0(x) + \Lambda T_1(x) + \Lambda^2 T_2(x) + \dots \end{aligned}\right\} \tag{3·1}$$

and the zero-order approximation gives the fully frozen result

$$\left.\begin{aligned} \rho_0 &= T_0^{\,1/(\gamma-1)}, \\ u_0 &= \left[u_\infty^2 + \frac{2\gamma}{\gamma-1}(1-T_0)\right]^{\frac{1}{2}}; \end{aligned}\right\} \tag{3·2 a}$$

T_0 is defined as a function of x by

$$T_0^{\,1/(\gamma-1)}\left[u_\infty^2 + \frac{2\gamma}{\gamma-1}(1-T_0)\right]^{\frac{1}{2}} = \frac{m}{A(x)}. \tag{3·2 b}$$

Corresponding first-order solutions are, for example,

$$\sigma_1 = \int_{x_\infty}^{x} F_0(s)\,[\bar{\sigma}_0(s) - \sigma_\infty]\,\mathrm{d}s, \tag{3·3 a}$$

$$\left(1 - \frac{\gamma T_0}{u_0^2}\right)\frac{T_1}{T_0} = (\gamma-1)\frac{\sigma_1}{u_0^2} - (\gamma-1)\int_{x_\infty}^{x} \frac{F_0(s)}{T_0(s)}\,[\bar{\sigma}_0(s) - \sigma_\infty]\,\mathrm{d}s, \tag{3·3 b}$$

where $\bar{\sigma}_0 = \bar{\sigma}(T_0)$,

$$F_0 = \rho_0\,\Omega(T_0)/u_0, \tag{3·4}$$

and x_∞ denotes the initial value of x. For a convergent-divergent nozzle, for which $u_\infty = 0$ and $\sigma_\infty = \bar{\sigma}_\infty$, x_∞ is taken as $-\infty$.

If x_∞ is finite, $u_\infty \neq 0$, and the mass flow rate is known and independent of Λ (such a case would be supersonic flow through a divergent nozzle), then the expansion defined by (3·1) does indeed represent a valid solution for $x = O(1)$ and equations (3·2) and (3·3) give the leading terms in this solution. For this solution to be valid for all x requires a constraint on the minimum rate of growth of the nozzle (see below and part II).

If the flow is initially subsonic this solution is singular at the geometric throat ($u_0^2 = \gamma T_0$), but, for $u_\infty \neq 0$, the solution is easily rendered uniformly valid, for $x = O(1)$, by taking into account the dependence of the mass flow on the rate parameter. The mass flow m is strictly defined by conditions at the sonic point ($u = \surd(\gamma T)$) which, for $\Lambda \neq 0$, lies downstream of the throat. As $\Lambda \to 0$ the distance between the sonic point and the geometric throat becomes 'small' and it is sufficient to make the solution well behaved at the geometric throat. It follows from (3·1) that $m(\Lambda)$ should also be expanded in integral powers of Λ if $u_\infty \neq 0$.

A similar scheme for $u_\infty = 0$ is outlined in detail in § 4 and the reader is referred to that section for a practical application of the approach.

However, a more serious restriction than the difficulty at the throat arises when $u_\infty = 0$. For such flows the integrals defined in (3·3 a) and (3·3 b) do not necessarily converge as $x \to -\infty$. If the nozzle shape is described asymptotically by

$$A \sim |x|^n \sum_{r=0} a_r |x|^{-r}, \tag{3·5}$$

where $n > 0$, then, apart from some constant factor, (3·3 a) indicates that

$$\mathrm{d}\sigma_1/\mathrm{d}x \sim -|x|^{-n} \quad (n \neq 1)$$

as $x \to -\infty$. Consequently σ_1 remains bounded only if $n > 1$. Moreover, the restriction $n > 1$ merely postpones the occurrence of any singular behaviour to a later stage. It is easily shown that the corresponding expression for σ_2 is singular, at $-\infty$, for all values of $n > 0$.

This mathematical difficulty obviously arises because the local value of the rate parameter, as opposed to its characteristic value, becomes large in the reservoir (stagnation) region, thus violating the near-frozen assumption. A method of obtaining a valid solution within this region is described in § 4. It is also shown in § 5 that some account has to be taken of the behaviour in the reservoir zone in constructing a valid solution for $x = O(1)$.

In addition to this singular behaviour as $x \to -\infty$, there is a possible further non-uniformity in conventional near-frozen solutions which is associated with the asymptotic nozzle shape far downstream. For nozzles such that

$$A \sim x^\nu \sum_{r=0} A_r x^{-r} \tag{3·6}$$

as $x \to \infty$, the integrals defined in (3·3) do not necessarily converge when the relaxation frequency is defined by (2·9), or more generally when the relaxation frequency decays as some power of the temperature in the limit $T \to 0$. It is apparent, from the fully frozen solution, that the integral defining σ_1 will converge only if

$$\nu > \frac{1}{1+(\gamma-1)\delta}, \tag{3·7}$$

which is equivalent to asking not only that $\tau \to \infty$ but also that

$$\int^x \frac{\mathrm{d}x}{\tau_0}$$

is bounded. One might expect that the divergence of this integral will correspond to an eventual decay of the vibrational energy towards its local equilibrium value, which at large distances is presumably ground level. However, the condition governing the convergence of the integral defining T_1/T_0 is even more restrictive. In this case

$$\nu > \frac{1}{2-\gamma+(\gamma-1)\delta} \tag{3·8}$$

in order that

$$\int^{\sigma_1(x)} \frac{\mathrm{d}\sigma_1}{T_0}$$

is bounded for large x. This integral can be regarded as an entropy-producing term and a similar quantity has been called the pseudo-entropy (Broer 1951). The implications of (3·7) and (3·8) when these inequalities are not satisfied are discussed at length in part II.† In this paper it is assumed that (3·8) holds, thus automatically satisfying the inequality (3·7) for $\gamma > 1$, $\delta \geqslant 0$.

No difficulty arises for nozzle shapes which are asymptotically described by (3·6), if Ω decreases faster than any power of T. Consequently expressions of the type (2·8) always lead to the result that the vibrational energy freezes. If (3·8) is satisfied both (2·8) and (2·9) imply asymptotic frozen solutions. However, the magnitude of the frozen energy level will depend on the choice of temperature dependence. Since the various expressions all provide reasonable fits to the available experimental data this choice may not be important in practice, provided that no extrapolation below the temperatures at which the data is known is necessary. Even then, if the flow does freeze, the choice may still not be significant since the dominant contributions to, for example, the final frozen level come from regions where the temperature is not small. Note that these remarks hold only if (3·8) is satisfied (see part II).

4. The flow through a convergent–divergent nozzle—the reservoir zone

For $u_\infty = 0$ the conventional near-frozen solution is singular as $x \to -\infty$. In this stagnation region, referred to as the inner layer (see figure 1 (*b*)), it is necessary to seek an alternative solution. Downstream of this layer, where $x = O(1)$, some perturbation about the frozen solution should be valid and this will be referred to as the outer solution. The asymptotic nozzle shape, far upstream, belongs to the family defined in equation (3·5).

If
$$\epsilon = \bar{\sigma}_\infty - \sigma, \tag{4·1}$$
then, from equation (2·3),
$$1 - T = \frac{\gamma-1}{2\gamma}u^2 - \frac{\gamma-1}{\gamma}\epsilon, \tag{4·2}$$
and it is assumed that $1-T$, u^2 and ϵ are all of the same order of magnitude in the inner layer where u is 'small'. It follows from (2·2*b*) that in this region
$$1-\rho = \frac{u^2}{2\gamma} + \frac{\gamma-1}{\gamma}\epsilon + O(u^4). \tag{4·3}$$
If Ω and $\bar{\sigma}$ are expanded in Taylor series near $T = 1$, ($x \to -\infty$), the rate equation can be re-written in terms of ϵ and u as
$$\frac{d\epsilon}{dx} = \frac{-\Lambda}{u}\left[1 - \frac{1}{2\gamma}\{1+(\gamma-1)w_1\}u^2 - \frac{\gamma-1}{\gamma}(1-w_1)\epsilon\right]\left[\left\{1+c_1\frac{\gamma-1}{\gamma}\right\}\epsilon - c_1\frac{\gamma-1}{2\gamma}u^2 + c_2\left\{\frac{\gamma-1}{2\gamma}u^2 - \frac{\gamma-1}{\gamma}\epsilon\right\}\right] + O(\Lambda u^5), \tag{4·4}$$
where
$$w_n = \frac{1}{n!}\left(\frac{d^n\Omega}{dT^n}\right)_{T=1}, \tag{4·5}$$
$$c_n = \frac{1}{n!}\left(\frac{d^n\bar{\sigma}}{dT^n}\right)_{T=1}. \tag{4·6}$$

† The possible divergence of similar integrals has also been noted recently by Cheng & Lee (1966) for a dissociating gas.

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The continuity equation, together with (4·3), provides a second relation between ϵ and u which is

$$Au\left[1-\frac{\gamma-1}{\gamma}\epsilon-\frac{u^2}{2\gamma}+O(u^4)\right]=m(\Lambda). \tag{4·7}$$

From (3·5) it is apparent that

$$u\sim (m/a_0)\,|x|^{-n},$$

as $x\to-\infty$. Hence, with regard to the form of equation (4·4) a non-trivial scaling of the variables in the inner layer is

$$u=\Lambda^{n/(n+1)}U,\quad \epsilon=\Lambda^{2n/(n+1)}E,\quad x=\Lambda^{-1/(n+1)}X, \tag{4·8}$$

where the variables U, E and X are $O(1)$. The equations satisfied by E and U are

$$\frac{dE}{dX}+\frac{1}{U}\left[\left(1+c_1\frac{\gamma-1}{\gamma}\right)E-c_1\frac{\gamma-1}{2}U^2\right]$$
$$=\frac{\Lambda^{2n/(n+1)}}{U}[b_0E^2+b_1EU^2+b_2U^4]+O(\Lambda^{4n/(n+1)}), \tag{4·9}$$

where the constants b_0, b_1, and b_2 are defined in appendix I, and

$$|X|^nU[a_0+\Lambda^{1/(n+1)}a_1|X|^{-1}+\Lambda^{2/(n+1)}a_2|X|^{-2}]$$
$$=m(\Lambda)[1+\Lambda^{2n(n+1)}(\tfrac{1}{2}U^2+\{(\gamma-1)/\gamma\}E)]+(O(\Lambda^{4n/(n+1)}),\ O(\Lambda^{3/(n+1)})). \tag{4·10}$$

As yet $m=m\,(\Lambda)$ is unknown. Its variation with Λ can be found from conditions near the sonic point in the outer region (see § 5). It seems reasonable to suppose that to zero order it will be $O(1)$ and independent of Λ. This implies, as will be verified by the matching, that its zero-order value is that for fully frozen flow. The corresponding zero-order quantities U_0 and E_0 satisfy

$$U_0=\frac{m_0}{a_0}|X|^{-n}, \tag{4·11}$$

$$\frac{dE_0}{dX}+\frac{a_0}{m_0}\left\{1+c_1\frac{\gamma-1}{\gamma}\right\}|X|^n\,E_0=c_1\frac{\gamma-1}{2}\frac{m_0}{a_0}|X|^{-n}, \tag{4·12}$$

where m_0 is the zero-order approximation to $m(\Lambda)$. Not surprisingly the solution for the velocity, to this order, is uncoupled from the energy distribution, apart from the dependence on m_0, and is given solely by the asymptotic form of the continuity equation. Some simplification in the factors occurring in (4·12) is possible. If C_p is the specific heat at constant pressure, including the contribution $C_{vib.}$ from the vibrational mode, and C_{pa} is the corresponding specific heat including only the contribution from the active modes of translation and rotation, then

$$1+c_1\frac{\gamma-1}{\gamma}=\frac{C_{p,\infty}}{C_{pa}},\quad (\gamma-1)\,c_1=\frac{C_{vib.,\infty}}{C_{pa}} \tag{4·13}$$

It is convenient to define

$$G(\alpha,t)=e^t\int_t^\infty s^{\alpha-1}\,e^{-s}\,ds \tag{4·14}$$

(see appendix II). The solution of (4·12), satisfying the boundary condition $E_0\to 0$ as $X\to-\infty$, is

$$E_0=\frac{C}{n+1}k^{(n-1)/(n+1)}G\left(\frac{1-n}{1+n},Z\right), \tag{4·15}$$

where

$$C=\frac{1}{2}\frac{m_0}{a_0}\frac{C_{vib.,\infty}}{C_{pa}},\quad k=\frac{1}{n+1}\frac{a_0}{m_0}\frac{C_{p,\infty}}{C_{pa}},\quad Z=k|X|^{n+1}. \tag{4·16}$$

It is worth noting that the upstream limit, $X \to -\infty$, of (4·15) gives

$$E_0 \sim \frac{C}{(n+1)\,k}\left[|X|^{-2n} - \frac{2n}{n+1}\frac{1}{k}|X|^{-(1+3n)} + O(|X|^{-(2+4n)})\right],$$

which is consistent with the usual inverse power series expansion about stagnation conditions (Hall & Russo 1959).

Of more interest in the present context is the downstream limit $|X| \to 0$ of (4·15). The behaviour of the solution in this limit provides matching conditions for the upstream (inner) limit of the near frozen outer solution (see § 5). As $|X| \to 0$

$$E_0 \sim \frac{C}{n-1}\left[|X|^{-(n-1)} - k^{(n-1)/(n+1)}\Gamma\left(\frac{2}{n+1}\right) + \frac{n+1}{2}k\,|X|^2 - \Gamma\left(\frac{2}{n+1}\right)k^{2n/(n+1)} + O(|X|^{2n+2}, |X|^{n+3})\right] \qquad (4·17)$$

if $n \neq 1$. If $n = 1$

$$E_0 = C[-\ln|X| - \tfrac{1}{2}\kappa - k|X|^2\ln|X| - \tfrac{1}{2}k(\kappa-1)\,|X|^2 + O(|X|^4\ln|X|)], \qquad (4·18)$$

where

$$\kappa = \ln k + \gamma_e \qquad (4·19)$$

and γ_e is Euler's constant. In equation (4·17) the terms are written in descending order of magnitude for $n > 1$, and the magnitude of the error term depends on sgn $(n-1)$.

The inner solutions for ρ and T follow directly from equations (4·2) and (4·3). Thus, to first order

$$\left.\begin{aligned} 1 - T &= \Lambda^{2n/(n+1)}\frac{\gamma-1}{\gamma}(\tfrac{1}{2}U_0^2 - E_0), \\ 1 - \rho &= \Lambda^{2n/(n+1)}\frac{\gamma-1}{\gamma}\left(\frac{U_0^2}{2(\gamma-1)} + E_0\right), \end{aligned}\right\} \qquad (4·20)$$

where U_0 and E are given by (4·11) and (4·15).

5. First- and second-order solutions in the outer layer

5·1. *Discussion*

In the outer region x, ρ, T, and hence F are $O(1)$. Physically, it can still be expected that ϵ will be $o(1)$ since the behaviour in this region is primarily that of a near-frozen flow in which σ changes relatively slowly. From the form of the rate equation (2·4), expressed in outer variables as

$$\mathrm{d}\epsilon/\mathrm{d}x = \Lambda F(\bar{\sigma}_\infty - \bar{\sigma} - \epsilon), \qquad (5·1)$$

it might be presumed that ϵ is $O(\Lambda)$, but it will be seen that this is so only if $n > 1$. In terms of the outer variables equations (4·17) and (4·18) become

$$\epsilon \sim \frac{C}{n-1}\left[|x|^{-(n-1)} - \Lambda^{2n/(n+1)}k^{(n-1)/(n-1)}\Gamma\left(\frac{2}{n+1}\right) + \Lambda^2\frac{n+1}{2}k\,|x|^2 + O(\Lambda^{1+2n/(n+1)})\right] \qquad (5·2)$$

for $n \neq 1$. If $n = 1$

$$\epsilon \sim C[-\tfrac{1}{2}\Lambda\ln\Lambda - \Lambda(\ln|x| + \tfrac{1}{2}\kappa) - \tfrac{1}{2}\Lambda^2\ln\Lambda\,k\,|x|^2 - \Lambda^2(k|x|^2 - \tfrac{1}{2}k(1-\kappa)\,|x|^2) + O(\Lambda^3\ln\Lambda)]. \qquad (5·3)$$

In (5·2) the terms are arranged in descending order of magnitude for $n > 1$. The dominant term is then $O(\Lambda)$. If $n < 1$ the leading term is $O(\Lambda^{2n/(n+1)})$. Note that when $n = 1$ logarithmic terms occur and in particular the first term is $O(\Lambda \ln \Lambda)$.

These terms define the order of magnitude of ϵ in the outer region. Moreover, it can be shown that for $n \geqslant 1$ the first two terms in (5·2) and (5·3) correctly indicate the form of the expansion in the respective outer solutions.

Thus for $n > 1$ the outer expansion is

$$\epsilon = \Lambda \epsilon_1 + \Lambda^{2N} \epsilon_{2N} + \ldots, \tag{5·4}$$

where $$N = n/(n+1).$$

Similarly for $n = 1$

$$\epsilon = \Lambda \ln \Lambda \epsilon_{1,1} + \Lambda \epsilon_1 + \ldots. \tag{5·5}$$

For $n < 1$ only the first term is known, and

$$\epsilon = \Lambda^{2N} \epsilon_{2N} + \ldots. \tag{5·6}$$

The error terms in these expansions can be found once higher order terms in the inner layer are known (see § 6).

Terms are labelled ϵ_i corresponding to the power of Λ with which they are associated. A double suffix notation is employed for logarithmic terms; the second suffix denotes the power of the logarithm. (Difficulties arise with a conventional method of enumerating the terms since the sequence depends on n.) It is convenient to discuss separately the cases $n > 1$, $n = 1$, and $n < 1$.

5·2. *Solution for* $n > 1$

In conjunction with (5·4) the outer expansions for the variables ρ, u and T are

$$\rho = \rho_0(x) + \Lambda \rho_1(x) + \Lambda^{2N} \rho_{2N}(x) + \ldots, \text{ etc.} \tag{5·7}$$

It also follows that the mass flow can be expanded in a similar fashion,

$$m = m_0 + \Lambda m_1 + \Lambda^{2N} m_{2N} + \ldots. \tag{5·8}$$

The term m_0, which is $O(1)$, is identified with the zero-order approximation in the inner layer.

Substitution in equations (2·1) to (2·3) shows that the zero-order solution is the frozen solution given by equations (3·2), with $u_\infty = 0$, save that the first of the relations (3·2a) is replaced by

$$\rho_0 T_0^{-1/(\gamma-1)} = S_0, \tag{5·9}$$

where S_0 is an arbitrary constant. Equation (5·9) follows from the differential form of (2·2). Although it appears obvious that $S_0 = 1$, this does not immediately follow from the initial boundary condition at $-\infty$ since these are not necessarily the appropriate inner limits of this outer solution. However, matching with the downstream expansion of the inner solution for the density formally shows that

$$S_0 = 1. \tag{5·10}$$

(The only constant term, of $O(1)$, that arises from (4·20), together with (4·11) and (4·17), is unity.)

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From conditions at the throat, where the local flow speed is equal to the frozen sound speed ($u_0^2 = \gamma T_0$),

$$m_0 = \gamma \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)} \tag{5·11}$$

as expected.

The first-order departure from frozen flow, ϵ_1, satisfies

$$\mathrm{d}\epsilon_1/\mathrm{d}x = F_0(\bar{\sigma}_\infty - \bar{\sigma}_0), \tag{5·12}$$

where F_0 and $\bar{\sigma}_0$ are defined in (3·4). Since

$$F_0(\bar{\sigma}_\infty - \bar{\sigma}_0) = O(|x|^{-n})$$

as $x \to -\infty$, the solution of (5·12) can be written, for $n > 1$,

$$\epsilon_1 = K_1 + \int_{-\infty}^{x} F_0(s)[\bar{\sigma}_\infty - \bar{\sigma}_0(s)]\,\mathrm{d}s. \tag{5·13}$$

As $x \to -\infty$ $$\epsilon_1 \sim K_1 + \frac{C}{n-1}|x|^{1-n} + O(|x|^{-n}).$$

From the outer expansion of the inner solution (5·2) it is seen that terms $O(\Lambda)$ match if

$$K_1 = 0. \tag{5·14}$$

The corresponding perturbations to the flow variables are given by

$$\left.\begin{aligned}
&\frac{\rho_1}{\rho_0} + \frac{u_1}{u_0} = \frac{m_1}{m_0} = M_1,\\
&\frac{\rho_1}{\rho_0} - \frac{1}{\gamma-1}\frac{T_1}{T_0} = e_1 - \int_{-\infty}^{x} \frac{F_0(s)}{T_0(s)}(\bar{\sigma}_\infty - \bar{\sigma}_0(s))\,\mathrm{d}s = S_1(x),\\
&\frac{u_1}{u_0} + \frac{\gamma}{\gamma-1}\frac{T_1}{u_0^2} = \frac{\epsilon_1}{u_0^2} = \frac{\gamma T_0}{u_0^2}H_1(x),
\end{aligned}\right\} \tag{5·15}$$

where e_1 is an arbitrary constant and m_1, as yet, is unknown. These equations can be solved for ρ_1, u_1, and T_1, to give

$$\left.\begin{aligned}
&\left(1 - \frac{u_0^2}{\gamma T_0}\right)\frac{\rho_1}{\rho_0} = \frac{-u_0^2}{\gamma T_0}M_1 + S_1 + H_1,\\
&\left(1 - \frac{u_0^2}{\gamma T_0}\right)\frac{u_1}{u_0} = M_1 - S_1 - H_1,\\
&\left(1 - \frac{u_0^2}{\gamma T_0}\right)\frac{T_1}{T_0} = \frac{-(\gamma-1)u_0^2}{\gamma T_0}(M_1 - S_1) + (\gamma-1)H_1.
\end{aligned}\right\} \tag{5·16}$$

The solution is singular at the geometric throat, defined by $u_0^2 = \gamma T_0$, unless

$$M_1 = m_1/m_0 = S_1(x_t) + H_1(x_t), \tag{5·17}$$

where the suffix t denotes the throat position. In general, the critical point, defined by the frozen sonic point, will lie downstream of the geometric throat. For near-frozen flows, $\Lambda \to 0$, the distance of this point from the throat is $o(1)$ and as usual in perturbation solutions of this type it is sufficient to make the solution well behaved at the critical point defined by the zero-order solution.

Equation (5·17) provides only a single relationship between the unknowns m_1 and e_1 (which is implicitly contained in $S_1(x_t)$). Equation (5·16) can, however, be re-written, using (5·17), so that these constants are eliminated in the expressions for u_1 and T_1, but ρ_1 still contains the constant e_1. The inner limit, as $x \to -\infty$, of ρ_1 is, from (5·16), (5·15) and the zero-order solution,

$$\rho_1 \sim e_1 - \frac{\gamma-1}{\gamma}\frac{C}{n-1}|x|^{1-n} + \dots. \tag{5·18}$$

From the inner solution (4·20) the downstream limit of the inner solution is, from (4·11) and (4·17),

$$\rho \sim 1 - \frac{1}{2}\frac{m_0^2}{a_0^2}|x|^{-2n} - \Lambda\frac{\gamma-1}{\gamma}\frac{C}{n-1}|x|^{1-n} + \Lambda^{2n/(n+1)}\frac{\gamma-1}{\gamma}\frac{C}{n-1}k^{(n-1)/(n+1)}\Gamma\left(\frac{2}{n+1}\right) + O(\Lambda^2). \tag{5·19}$$

Matching terms $O(\Lambda)$ shows that

$$e_1 = 0. \tag{5·20}$$

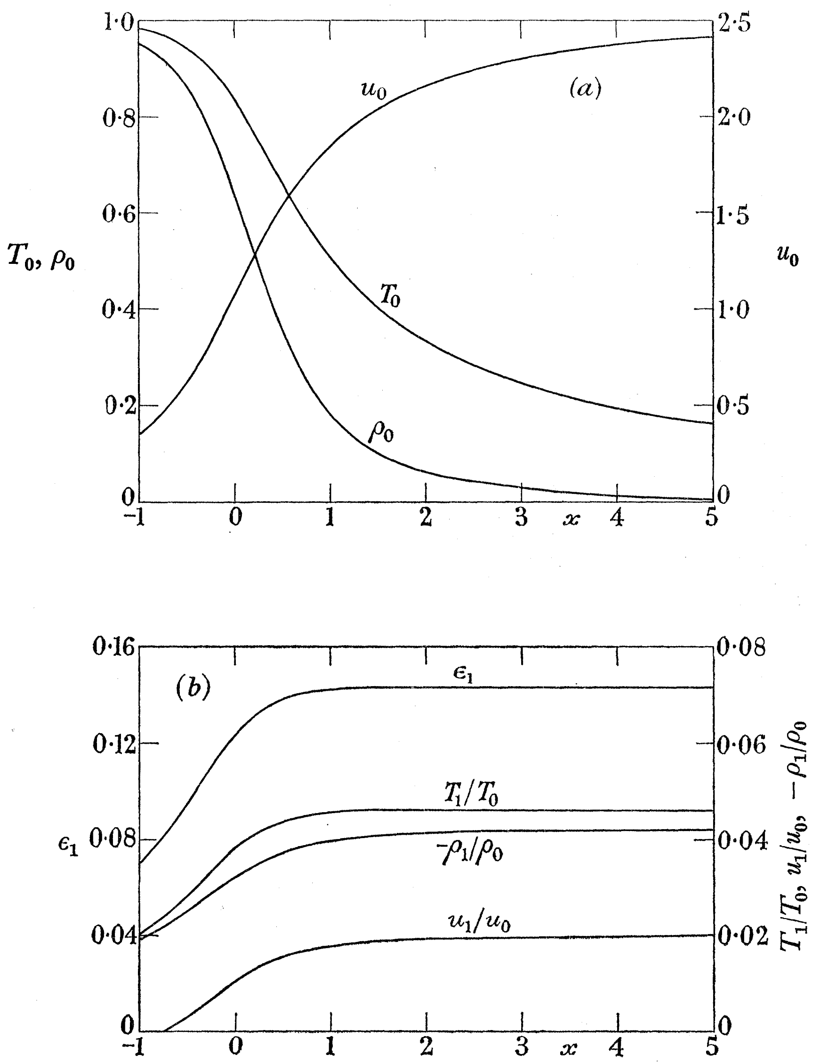

FIGURE 2. (a) Zero and (b) first-order solutions for flow through a hyperbolic nozzle, with $\theta_v = 1$ and $B = 7·394$.

As a numerical example the first-order perturbations have been evaluated for a diatomic gas in the particular case of flow through a hyperbolic nozzle, $A = 1+x^2$, with

$$\Omega = T\exp\{-B\theta_v^{\frac{1}{3}}(T^{-\frac{1}{3}}-1)\},$$

where $B = 7{\cdot}394$. This latter expression is taken from the paper by Phinney (1964) and provides a reasonable correlation of the experimental data for diatomic gases over a fairly broad temperature range.

The initial conditions for the curves of ϵ_1, ρ_1/ρ_0, etc., given in figure 2 (*b*) are such that $\theta_v = 1$. Figure 2 (*a*) shows the zero-order fully frozen solution and a combination of the two figures enables the absolute values ρ_1, etc., to be found.

In general higher terms will satisfy a similar set of relations to (5·13), (5·15) and (5·16), with ϵ_1, M_1, etc., replaced by ϵ_i, M_i, etc., where

$$\left.\begin{aligned} \mathrm{d}\epsilon_i/\mathrm{d}x &= f_i(u_{i-j}, T_{i-j}, \rho_{i-j}, \epsilon_{i-j}),\\ \text{or}\qquad \epsilon_i &= K_i+\mathscr{F}_i(x),\end{aligned}\right\} \tag{5·21}$$

and

$$\left.\begin{aligned} M_i &= m_i/m_0+\mathscr{M}_i(\rho_{i-j}, u_{i-j}),\\ S_i &= e_i+\mathscr{S}_i(\rho_{i-j}, T_{i-j}, \epsilon_i, \epsilon_{i-j}),\\ H_i &= H_i(u_{i-j}, \epsilon_i),\end{aligned}\right\} \tag{5·22}$$

where $j > 0$ and is not necessarily an integer. (Note that these quantities will not necessarily depend on all the u_{i-j}, etc., for all $j > 0$.) $\mathscr{F}_i$, $\mathscr{M}_i$, $\mathscr{S}_i$, and H_i can be considered as known functions of x. Matching with the outer limit of the inner solution for ϵ_i will determine K_i. The solution will be well behaved at the throat provided, as in (5·17),

$$M_i(x_t) = S_i(x_t)+H_i(x_t),$$

or

$$m_i/m_0+\mathscr{M}_i(x_t) = e_i+\mathscr{S}_i(x_t)+H_i(x_t), \tag{5·23}$$

which gives one relation between m_i and e_i. As for $i = 1$, the solution is uniquely determined by matching the inner and outer solutions for ρ. For the lower-order terms ρ is a linear function of E in the inner layer and a simple rule can be found for obtaining m_i in terms of K_i, $\mathscr{S}_i(x_t)$ and $H_i(x_t)$ (see below and § 7).

For terms which are generated in the near-frozen solution solely by the inner-layer behaviour, i.e. terms which are not obviously present from the form of the outer equations but which must be included in order to satisfy the matching conditions, the outer solution adopts a very simple form. The second-order perturbation E_{2N} belongs to this class, and

$$\mathscr{F}_{2N} = 0, \quad \text{or} \quad \epsilon_{2N} = K_{2N}. \tag{5·24}$$

Matching with (5·2) shows that

$$K_{2N} = \frac{-C}{n-1}k^{(n-1)/(n+1)}\Gamma\left(\frac{2}{n+1}\right). \tag{5·25}$$

The corresponding quantities $\mathscr{M}_{2N}$, etc., are

$$\mathscr{M}_{2N} = 0, \quad \mathscr{S}_{2N} = 0, \quad H_{2N} = K_{2N}/\gamma T_0(x). \tag{5·26}$$

From (5·23)

$$\frac{m_{2N}}{m_0} = e_{2N}+\frac{\gamma+1}{2\gamma}K_{2N}. \tag{5·27}$$

The inner limit for ρ_{2N} is

$$e_{2N}+K_{2N}/\gamma+O(|x|^{-2n})$$

(using (5·16) and replacing S_1 by S_{2N}, etc.). Matching with (5·20) gives

$$e_{2N}+\frac{K_{2N}}{\gamma}=\frac{\gamma-1}{\gamma}\frac{C}{n-1}k^{(n-1)/(n+1)}\Gamma\left(\frac{2}{1+n}\right),$$

or

$$e_{2N}=-K_{2N} \tag{5·28}$$

from (5·25). This simple result is a consequence of the fact that, to this order, ρ is a linear function of E in the inner layer. Hence

$$\frac{m_{2N}}{m_0}=-\frac{\gamma-1}{2\gamma}K_{2N}, \tag{5·29}$$

and

$$\frac{\rho_{2N}}{\rho_0}=-\frac{\gamma-1}{\gamma}K_{2N},\quad \frac{u_{2N}}{u_0}=\frac{\gamma-1}{2\gamma}K_{2N},\quad \frac{T_{2N}}{T_0}=\frac{\gamma-1}{\gamma}K_{2N}. \tag{5·30}$$

Note that ρ_{2N}, etc., are directly proportional to the fully frozen solutions ρ_0, etc.

For the particular numerical example discussed above ($\theta_v=1$, $n=2$)

$$K_{2N}=K_{4/3}=-0·1037.$$

5·3. *Solution for $n=1$*

For $n=1$ the appropriate outer expansions are

$$\rho=\rho_0+\Lambda\ln\Lambda\rho_{1,1}+\Lambda\rho_1+\ldots,\text{ etc.,} \tag{5·31}$$

and similarly for the mass flow. The zero-order outer solution is again easily shown to be the fully frozen flow. For the first-order term, $O(\Lambda\ln\Lambda)$, the solution closely parallels that outlined for the second-order term when $n>1$, since

$$\mathscr{F}_{1,1}=0. \tag{5·32}$$

Matching with the outer expansion of the inner solution (5·3) shows that

$$e_{1,1}=K_{1,1}-\tfrac{1}{2}C. \tag{5·33}$$

The solution for the flow variables obviously adopts a similar form to that already outlined for the second-order solution when $n>1$. In fact since ρ is again a linear function of E, to this order, within the inner layer

$$e_{1,1}=-K_{1,1},\quad \frac{m_{1,1}}{m_0}=-\frac{\gamma-1}{2\gamma}K_{1,1}. \tag{5·34}$$

The full solution follows from (5·30) with K_{2N} replaced by $K_{1,1}$.

For the terms $O(\Lambda)$

$$\mathscr{F}_1=F_0(\bar{\sigma}_\infty-\bar{\sigma}_0), \tag{5·35}$$

which is formally the same expression as for $n>1$. However, for $n=1$

$$\mathscr{F}_1\sim C|x|^{-1}[1+O(|x|^{-1})]$$

as $x\to-\infty$ and e_1 is not bounded in this limit. This singular behaviour in what corresponds to the normal first-order term in a conventional near-frozen solution is, in effect, predicted by the occurrence of the term $O(\Lambda \ln \Lambda)$. The solution for ϵ_1 can be written

$$\epsilon_1 = K_1 + \int_{-\infty}^{x} \left\{\mathscr{F}_1(s) - \frac{C}{1+|s|}\right\} \mathrm{d}s + \operatorname{sgn} x \ln\{1+|x|\}. \tag{5·36}$$

The factor $(1+|x|)^{-1}$ has been used, rather than $|x|^{-1}$, to avoid any difficulties at $x = 0$. Matching with the outer expansion of the inner solution (5·3) gives

$$K_1 = -\tfrac{1}{2}\kappa C. \tag{5·37}$$

The flow variables are implicitly defined by

$$\left.\begin{aligned} \mathscr{M}_1(x) &= 0 \\ \mathscr{S}_1(x) &= -\int_{-\infty}^{x} \left\{\frac{\mathscr{F}_1(s)}{T_0(s)} - \frac{C}{1+|s|}\right\} - \operatorname{sgn} x \ln(1+|x|), \\ H_1(x) &= \frac{\epsilon_1(x)}{\gamma T_0(x)}. \end{aligned}\right\} \tag{5·38}$$

Matching the inner and outer solutions for ρ gives

$$e_1 = -K_1 \tag{5·39}$$

as usual. Hence

$$m_1/m_0 = \mathscr{S}_1(x_t) + H_1(x_t) - K_1. \tag{5·40}$$

5·4. *Solution for* $n < 1$

Again the zero-order solution is the fully frozen one. The first order correction, $O(\Lambda^{2N})$, is easily shown to be that given by (5·25) and (5·30) in § 5·2. It is not strictly possible, at this stage, to obtain the next term for $n < 1$. Higher order approximations in the inner layer show that this term, not surprisingly, is $O(\Lambda)$ (see § 6).

The separation of the solutions for $n \gtrless 1$ is convenient but somewhat artificial. Moreover, the case $n = 1$ is apparently only a particular example of a general property of the expansion. From the outer expansion of the inner solution it appears that logarithmic terms may eventually be generated for each rational value of n (see § 8). These distinctions, other than for $n = 1$, will not appear until a later stage in the analysis. Certain logarithmic terms will, however, arise for all values of n (see § 7).

6. Higher approximations in the inner layer

6·1. $n > 1$

For simplicity, particular attention is given to the case $n > 1$. Some comments on the solution when $n \leqslant 1$ are made at the end of this section.

Since, for $n > 1$

$$m = m_0 + \Lambda m_1 + \Lambda^{2N} m_{2N} + \ldots,$$

it follows from the form of the inner equations (4·9) and (4·10) that in this case

$$U = U_0 + \Lambda^{1-N} U_{1-N} + \Lambda^{2(1-N)} U_{2-2N} + O(\Lambda^{3(1-N)}, \Lambda), \tag{6·1}$$

and similarly for E. Note that the magnitude of the error term depends on whether $n \gtrless 2$.

The solutions for U_{1-N} and U_{2-2N} are still uncoupled from the energy equation (see (4·10)). Indeed

$$U = m/(A\Lambda^N) + O(\Lambda^{2N}) \tag{6·2}$$

and the solution for the energy distribution E, correct to $O(\Lambda^{2(1-N)})$, could be obtained by solving equation (4·9), ignoring all the terms on the right-hand side, and replacing U by (6·2). Alternatively one can proceed systematically, as here, and deduce E_{1-N} and E_{2-2N} directly from the results

$$U_{1-N} = -\frac{m_0}{a_0}\frac{a_1}{a_0}|X|^{-(n+1)}, \tag{6·3}$$

$$U_{2-2N} = \frac{m_0}{a_0}\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)|X|^{-(n+2)}. \tag{6·4}$$

The equations satisfied by higher terms in E are of the form

$$\mathscr{D}E_i = L_i(E_{i-j}, U_{i-j}),$$

where

$$\mathscr{D} \equiv \frac{\mathrm{d}}{\mathrm{d}X} + \frac{\mathrm{d}Z}{\mathrm{d}|X|}$$

and $j > 0$. In particular

$$L_{1-N} = -\frac{a_1}{a_0}(n+1)\,k\,|X|^{n-1}E_0 + C|X|^{-(n+1)}, \tag{6·5a}$$

$$L_{2-2N} = -\frac{a_1}{a_0}(n+1)\,k\,|X|^{n-1}E_{1-N} - \frac{a_2}{a_0}(n+1)\,k\,|X|^{n-2}E_0 + C\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)|X|^{-(n+2)}. \tag{6·5b}$$

The corresponding solution for E_{1-N} and E_{2-2N}, satisfying the boundary condition $E_i \to 0$ as $X \to -\infty$, are

$$E_{1-N} = \frac{-2a_1}{a_0}\frac{C}{n+1}k^{n/(n+1)}Z^{n/(n+1)}G\left(\frac{-2n}{n+1}, Z\right), \tag{6·6}$$

$$E_{2-2N} = \frac{kC}{n(n^2-1)}\left[\left\{n(n-1)\frac{a_1^2}{a_0^2} + 2n\frac{a_2}{a_0}\right\}Z^{-1} + \left\{(n-1)\frac{a_1^2}{a_0^2} - 2n\frac{a_2}{a_0}\right\}G(0,Z) - Z^{(n-1)/(n+1)}\left\{2n^2\frac{a_2}{a_0} + (n^2-1)\frac{a_1^2}{a_0^2}Z\right\}G(-2N,Z)\right]. \tag{6·7}$$

6·2. *Other values of n*

If $n = 1$

$$m = m_0 + \Lambda \ln \Lambda m_{1,1} + \Lambda m_1 + \ldots$$

and a logarithmic term is now fed into the inner expansion. It follows from (4·10) that the inner expansion for U is

$$U = U_0 + \Lambda^{\frac{1}{2}}U_{\frac{1}{2}} + \Lambda \ln \Lambda U_{11} + \Lambda U_1 + \ldots \tag{6·8}$$

and similarly for E. The error term depends on higher order terms in the outer solution.

$U_{\frac{1}{2}}$ and $E_{\frac{1}{2}}$ are still given by equations (6·3) and (6·6) with $n = 1$, though the asymptotic expansion for $|X| \to 0$ will differ. (This expansion is easily found from (6·6) in this special case.) The higher order terms $E_{1,1}$, etc., can also be derived without difficulty, but for lack of space are not set down here.

When $n < 1$

$$m = m_0 + \Lambda^{2N} m_{2N} + \dots$$

and the appropriate inner expansion is

$$U = U_0 + \Lambda^{1-N} U_{1-N} + \Lambda^{2N} U_{2N} + \dots,$$

if $1 > n > \frac{1}{2}$. If $n < \frac{1}{2}$ the dominant first-order term is $O(\Lambda^{2N})$, which implies that it may be necessary to divide the range of n into still smaller subdivisions (see, however, § 5·4 and § 8). It does not seem worthwhile to pursue the precise form of the solution in these cases: in principle, at least, the approach is the same as for $n > 1$. A further and perhaps more cogent reason is that the algebra involved in computing the term U_{2N} is immense.

7. Higher approximations in the outer layer, $n > 1$.

Similar arguments to those advanced in § 5·2 show, using the second-order inner solution, that the outer solution can now be found correct to $O(\Lambda^2)$. From equations (4·8), (4·17), (6·6), (6·7) and appendix II the outer expansion of the inner solution for ϵ is

$$C^{-1}\epsilon \sim \Lambda\left\{\frac{|x|^{-(n-1)}}{n-1} - \frac{a_1}{a_0 n}|x|^{-n} + \frac{1}{n+1}\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)|x|^{-(n+1)} + \dots\right\}$$
$$+\Lambda^{2n/(n+1)}\left\{-\frac{1}{n+1}k^{(n-1)/(n+1)}\Gamma\left(\frac{2}{n+1}\right)\right\} + \Lambda^2 \ln\Lambda\left\{-\frac{k}{n+1}\left(\frac{1}{n}\frac{a_1^2}{a_0^2} - \frac{2}{n-1}\frac{a_2}{a_0}\right)\right\}$$
$$+\Lambda^2\left\{\frac{n+1}{2(n-1)}k|x|^2 + \frac{a_1}{a_0 n}\left(\frac{n+1}{n-1}\right)k|x| - k\left(\frac{1}{n}\frac{a_1^2}{a_0^2} - \frac{2}{n-1}\frac{a_2}{a_0}\right)\right.$$
$$\left.\times \ln|x| - \frac{1}{n+1}d_{2-2N} + \dots\right\} + o(\Lambda^2) \tag{7·1}$$

where
$$d_{2-2N} = -k\left[\left\{\frac{1}{n}\frac{a_1^2}{a_0^2} - \frac{2}{n-1}\frac{a_2}{a_0}\right\}\kappa + \frac{(n+1)^2}{2}\left\{\frac{1}{n^2}\frac{a_1^2}{a_0^2} - \frac{2}{(n-1)^2}\frac{a_2}{a_0}\right\}\right]. \tag{7·2}$$

The most striking feature of this result is the occurrence of the term $O(\Lambda^2 \ln \Lambda)$. This logarithmic factor arises for any value of n, as opposed to those generated by particular rational values of n (for example, the term $O(\Lambda \ln \Lambda)$ which occurs in the expansion for $n = 1$).

It follows that the outer expansions now have the form

$$\rho = \rho_0 + \Lambda\rho_1 + \Lambda^{2N}\rho_{2N} + \Lambda^2 \ln\Lambda\rho_{2,1} + \Lambda^2\rho_2 + \dots, \text{ etc.}$$

From the outer form of the rate equation $\epsilon_{2,1}$ is a constant and matching with (7·1) gives

$$K_{2,1} = -\frac{Ck}{n+1}\left\{\frac{1}{n}\frac{a_1^2}{a_0^2} - \frac{2}{n-1}\frac{a_2}{a_0}\right\}. \tag{7·3}$$

It follows as usual that

$$e_{2,1} = -K_{2,1}, \tag{7·4}$$

and the solution for $\rho_{2,1}$, etc., is given by (5·30) with K_{2N} replaced by $K_{2,1}$. For the example outlined in § 5

$$K_{2,1} = 0·0369.$$

For the terms $O(\Lambda^2)$ the algebra is somewhat more lengthy. The details are summarized below. If

$$\Omega_1 = T_1\left(\frac{d\Omega}{dT}\right)_{T=T_0}, \quad \bar{\sigma}_1 = T_1\left(\frac{d\bar{\sigma}}{dT}\right)_{T=T_0}, \tag{7·5}$$

then

$$f_2 = F_0\left[\left(\frac{\rho_1}{\rho_0}+\frac{\Omega_1}{\Omega_0}-\frac{u_1}{u_0}\right)(\bar{\sigma}_2-\bar{\sigma}_0)-\epsilon_1-\bar{\sigma}_1\right], \tag{7·6}$$

and it is convenient to write

$$\mathscr{F}_2 = \int_{-\infty}^{x}\left\{f_2(s)+\frac{dg_2}{d|s|}\right\}ds - \operatorname{sgn} x\, g_2(x), \tag{7·7}$$

where

$$g_2(x) = \frac{n+1}{n-1}Ck\left[\tfrac{1}{2}|x|^2+\frac{a_1}{na_0}|x|-\frac{1}{n+1}\left\{\frac{n-1}{n}\frac{a_1^2}{a_0^2}-\frac{2a_2}{a_0}\right\}\ln(1+|x|)\right]. \tag{7·8}$$

The solution for ϵ_2 then matches with (7·1) provided

$$K_2 = -\{C/(n+1)\}\,d_{2-2N}. \tag{7·9}$$

Similarly, the results for $\mathscr{M}_i$, etc., are

$$\left.\begin{aligned}
\mathscr{M}_2 &= -\rho_1 u_1/\rho_0 u_0,\\
\mathscr{S}_2 &= \frac{1}{2}\left(\frac{\rho_1}{\rho_0}\right)^2 - \frac{1}{2(\gamma-1)}\left(\frac{T_1}{T_0}\right)^2 + \int_{-\infty}^{x}\frac{T_1}{T_0^2}\frac{d\epsilon_1}{ds}\,ds\\
&\quad - \int_{-\infty}^{x}\left(\frac{f_2}{T_0}+\frac{dg_2}{d|s|}\right)ds + \operatorname{sgn} x\, g_2(x),\\
H_2 &= \frac{1}{\gamma}\frac{\epsilon_2}{T_0} - \frac{1}{2\gamma}\frac{u_1^2}{T_0},
\end{aligned}\right\} \tag{7·10}$$

which together with

$$\left.\begin{aligned}
e_2 &= -K_2,\\
m_2/m_0 &= \mathscr{S}_2(x_t)+H_2(x_t)-\mathscr{M}_2(x_t)-K_2,
\end{aligned}\right\} \tag{7·11}$$

define the solution for ρ_2, etc.

The second order solution is shown in figure 3 for the particular case discussed earlier in § 5.

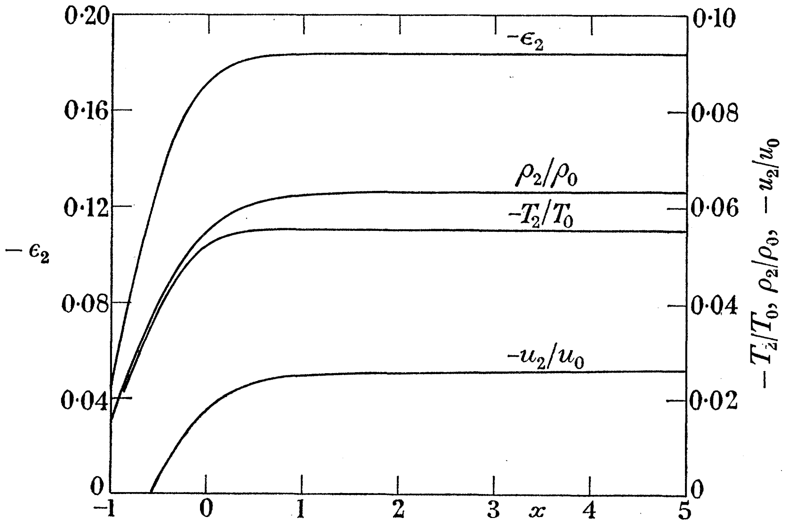

FIGURE 3. Second-order solutions for flow through a hyperbolic nozzle, with $\theta_v = 1$ and $B = 7·394$.

8. Discussion

It has been shown that the solution away from the reservoir zone for near-frozen flow ($\Lambda \to 0$) through a convergent-divergent nozzle can be written, correct to $O(\Lambda^2)$,

$$\epsilon = \Lambda\epsilon_1 + \Lambda^{2N}\epsilon_{2N} + \Lambda^2 \ln \Lambda \epsilon_{2,1} + \Lambda^2\epsilon_2 + \ldots, \text{ etc.,} \tag{8·1}$$

where $N = n(n+1)$, $n > 1$, and the asymptotic nozzle shape far upstream is described by

$$A \sim |x|^n.$$

Although it was obvious that a conventional near-frozen solution (expansion in integral powers of Λ) could not remain valid within the reservoir zone, less apparent was the striking influence that this domain could have on the outer expansion (8·1). The terms $O(\Lambda^{2N})$ and $O(\Lambda^2 \ln \Lambda)$ are generated solely by this region. If $n < 1$ the term $O(\Lambda^{2N})$ is then the dominant one on the right-hand side of (8·1) and in this case the effect of the reservoir zone provides the major deviation from a fully frozen flow.

Extension of the analysis to obtain higher order terms in the outer solution for $n \leqslant 1$ appears to offer no formal difficulties, but some care must be taken. Inspection of the outer expansion of the inner solution suggests that logarithmic terms may in fact arise for all rational values of n. If $n > 1$ it can be conjectured, for example, that logarithmic terms in x, and thus implicitly in Λ, will occur if, with the use of (7·1),

$$-(n-1) + i(n+1) - j = 0. \tag{8·2}$$

Here i and j are positive integers and the logarithmic term associated with this zero is $O(\Lambda^{i+1} \ln \Lambda)$. Equation (8·2) certainly includes all rational values of $n > 1$. Note in particular that the term $O(\Lambda^2 \ln \Lambda)$ corresponds to the solution $i = 1, j = 2$ which is independent of n. Similar remarks can be made for $n < 1$ by examining the equivalent expansion to (7·1). In that case, however, it does not necessarily follow that the only logarithmic term which is of a greater magnitude than Λ^2 is $O(\Lambda^2 \ln \Lambda)$. (See, for example, the expansion given for $n = 1$ in § 5.)

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Appendix I. The constants b_i

$$b_0 = \frac{\gamma-1}{\gamma}\left[1 - w_1 + \frac{\gamma-1}{\gamma}(1-w_1)\,c_1 - \frac{\gamma-1}{\gamma}c_2\right],$$

$$b_1 = \tfrac{1}{2}\{1 + (\gamma-1)\,w_1\} + \frac{\gamma-1}{\gamma}\{(\gamma-1)\,(2w_1-1)+1\}c_1 + \frac{(\gamma-1)^2}{\gamma}c_2,$$

$$b_2 = -\tfrac{1}{4}(\gamma-1)\,[\{1+(\gamma-1)\,w_1\}c_1 + (\gamma-1)\,c_2].$$

Appendix II. The function $G(\alpha, t)$

$$G(\alpha,t) = e^t \int_t^\infty s^{\alpha-1} e^{-s} \, ds,$$

is defined for all values of α, $t > 0$. If $\alpha > 0$ obviously

$$G(\alpha,t) = e^t[\Gamma(\alpha) - \gamma(\alpha,t)],$$

where $\gamma(\alpha, t)$ is the incomplete Γ function.

For non-integral values of $\alpha < 0$, repeated integration by parts gives

$$G(\alpha,t) = -\sum_{r=0}^{n} \frac{t^{\alpha+r}}{\alpha(\alpha+1)\dots(\alpha+r)} + e^t \left[\frac{\Gamma(\alpha+n+1) - \gamma(\alpha+n+1,t)}{\alpha(\alpha+1)\dots(\alpha+n)}\right]$$

where, here, n is any integer such that $\alpha+n+1 > 0$. The asymptotic expansion, $t \to 0$, follows trivially from this expression.

If α is a negative integer, then

$$G(\alpha,t) = -\sum_{r=0}^{\alpha_1-1} \frac{(-1)^{r+1} t^{r-\alpha_1}}{\alpha_1(\alpha_1-1)\dots(\alpha_1-r)} + \frac{(-1)^{\alpha_1} e^t \mathrm{Ei}(t)}{\alpha_1(\alpha_1-1)\dots 1},$$

where $\alpha_1 = -\alpha$ and $\mathrm{Ei}(t)$ is the exponential integral. In this case the asymptotic expansion for small t is,

$$-\sum_{r=0}^{\alpha_1-1} \frac{(-1)^{r+1} t^{r-\alpha_1}}{\alpha_1(\alpha_1-1)\dots(\alpha_1-r)} - \frac{(-1)^{\alpha_1} \ln t}{\alpha_1(\alpha_1-1)\dots 1} - \frac{(-1)^{\alpha_1}\gamma_e}{\alpha_1(\alpha_1-1)\dots 1} + O(t \ln t),$$

where γ_e is Euler's constant.

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